

A Non-Spectral Dense Banach Subalgebra of the Irrational Rotation Algebra

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Abstract

We give an example of a dense, simple, unital Banach subalgebra A of the irrational rotation C^* -algebra B , such that A is not a spectral subalgebra of B . This answers a question posed in T.W. Palmer's paper [1].

If A is a subalgebra of an algebra B (both algebras over the complex numbers), we say that A is a *spectral subalgebra* of B if the quasi-invertible elements of A are precisely the quasi-invertible elements of B which lie in A . In the language of [3], this is equivalent to saying that A is a *spectral invariant subalgebra* of B .

There are many known examples of dense unital Banach subalgebras of C^* -algebras which are not spectral. For example, see Example 3.1 of [3]. The example we give here is of interest because the Banach algebra is simple, and thus answers Question 5.12 of [1] in the negative.

Recall that the irrational rotation algebra associated with an irrational real number θ is the C^* -crossed product of the integers \mathbf{Z} with the commutative C^* -algebra of continuous functions on the circle $C(\mathbf{T})$, where $n \in \mathbf{Z}$ acts via $\alpha_n(\varphi)(z) = \varphi(z - n\theta)$, for $\varphi \in C(\mathbf{T})$ and $z \in \mathbf{T}$. Let $B = C^*(\mathbf{Z}, C(\mathbf{T}), \theta)$ denote this crossed product.

Let A be the set of functions F from \mathbf{Z} to $C(\mathbf{T})$ which satisfy the integrability condition

$$\|F\|_A = \sum_{n \in \mathbf{Z}} e^{|n|} \|F(n)\|_\infty < \infty,$$

where $\|\cdot\|_\infty$ denotes the sup norm on $C(\mathbf{T})$. Then A is complete for the norm $\|\cdot\|_A$ and is a Banach algebra. The algebra A is contained in $L^1(\mathbf{Z}, C(\mathbf{T}))$ with dense and continuous

inclusion, and hence contained in B with dense and continuous inclusion. Recall that the multiplication (in both A and B) is given by

$$F * G(n, z) = \sum_{m \in \mathbf{Z}} F(n, z) G(n - m, z - m\theta), \quad F, G \in A, \quad n \in \mathbf{Z}, \quad z \in \mathbf{T}.$$

Let $u_n = \delta_n \otimes 1 \in A$ denote the delta function at $n \in \mathbf{Z}$ tensored with the identity in $C(\mathbf{T})$. Then u_0 is the unit in both A and B .

Theorem 1 *The Banach algebra A is simple.*

Proof: We imitate the argument of [2]. Define a continuous linear map $P: A \rightarrow C(\mathbf{T}) \subseteq \mathbf{A}$ by $P(F) = F(0)$. Note that $\|P(F)\|_A \leq \|F\|_A$ for $F \in A$. Let J be a closed two-sided ideal in A , which is not equal to A . Since \mathbf{Z} acts ergodically on \mathbf{T} , we know that $C(\mathbf{T})$ has no nontrivial closed \mathbf{Z} -invariant ideals. Hence $J \cap C(\mathbf{T}) = \mathbf{0}$.

We show that $P(J) = 0$. It suffices to show that $P(J) \subseteq J$. Let $\epsilon > 0$ and $F \in A$. Let N be a sufficiently large integer for which

$$\sum_{|n| > N} e^{|n|} \|F(n)\|_\infty < \epsilon.$$

Define $F_1 \in A$ by $F_1(n) = 0$ if $|n| > N$, and $F_1(n) = F(n)$ for $|n| \leq N$. By the proof of Lemma 6 of [2], there exists unimodular functions $\theta_1, \dots, \theta_M \in C(\mathbf{T})$ such that

$$P(F_1) = \frac{1}{M} \sum_{n=1}^M \theta_n^* F_1 \theta_n.$$

(Here unimodular means that $|\theta_i(z)| = 1$ for each $z \in \mathbf{T}$ and $i = 1, \dots, M$.) Hence

$$\|P(F) - \frac{1}{M} \sum_{n=1}^M \theta_n^* F \theta_n\|_A \leq \|P(F - F_1)\|_A + \|F - F_1\|_A < 2\epsilon. \quad (*)$$

Now if $F \in J$, (*) shows that $P(F)$ can be approximated arbitrarily closely by elements of J . Since J is closed, this shows that $P(F) \in J$. Hence $P(J) \subseteq J$ and $P(J) = 0$.

If $P(Fu_n) = 0$ for all n , then $F(n) = 0$ for all n and so $F = 0$. Since J is a two-sided ideal and $P(J) = 0$, we have $P(Ju_n) = 0$ for all n . Hence $J = 0$ and A is simple. **Q.E.D.**

Theorem 2 *The Banach algebra A is not a spectral subalgebra of B .*

Proof: We construct an algebraically irreducible A -module which is not contained in any $*$ -representation of B on a Hilbert space. By Corollary 1.5 of [3], it will follow that A is not a spectral subalgebra of B .

Let E be the Banach A -module $C(\mathbf{T})$ with sup norm, and with (continuous) action of A given by

$$(F\varphi)(z) = \sum_n F(n, z) e^n \varphi(z - n\theta), \quad \varphi \in E, \quad F \in A, \quad z \in \mathbf{T}.$$

We show that E is in fact algebraically irreducible. Let $\varphi \in E$ be not identically equal to zero. Since the complex conjugate of φ is in A , the algebraic span $A\varphi$ contains $|\varphi|^2$, which we denote by ψ . Note $u_n\psi(z) = e^n\psi(z - n\theta)$. Since θ is irrational and \mathbf{T} is compact, there exists finitely many $n_1, \dots, n_k \in \mathbf{Z}$ such that the sum of $u_{n_i}\psi$ from $i = 1$ to k never vanishes on \mathbf{T} . If χ is this sum, then $1/\chi$ is in $C(\mathbf{T}) \subseteq \mathbf{A}$ so $1 \in A\varphi$ and hence $E = A\varphi$. This proves that E is algebraically irreducible.

It remains to show that no $*$ -representation of B on a Hilbert space contains E . But the action of \mathbf{Z} on $1 \in E$ is given by $u_n 1 = e^n 1$. Clearly the Hilbert space could not have a unitary, or even isometric, action of \mathbf{Z} . **Q.E.D.**

References

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